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# ON THE STRONG NO LOOP CONJECTURE FOR ARTINIAN RINGS WITH RADICAL CUBED ZERO

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ABSTRACT. This manuscript aims to prove the strong no-loop conjecture for an Artinian ring whose cube of its Jacobson radical is zero and under the condition that any simple module over this ring of finite projective dimension has a radical square zero of the cover projective of its first syzygy, to do this, we use a property of the simple module which realizes the minimum of the finite projective dimensions of simple modules.

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## 1. INTRODUCTION

Let $A$ be an artinian ring of which $J$ is its Jacobson radical and $mod(A)$ is the category of left $A$-modules of finite type. An important invariant of $A$ is its global dimension denoted $gdim(A)$, which is the supremum of projective dimensions of left $A$-modules of finite type. It is known that $gdim(A)$ is the supremum of the projective dimensions of simple $A$-modules and the number of non-isomorphic simple $A$-modules is finite and also $J$ is nilpotent see [5, Proposition 3.1].

We define the quiver (that's to say a directed graph) of extensions of $A$, whose set of vertices is a complete set of representatives of isomorphism classes of simple $A$-modules, and given two vertices $S; T$ there is an arrow from $S$ to $T$ if the extension group $Ext^1_A(S,T)$ is not zero. The no-loop conjecture says that if $gdim(A)$ is finite, then the quiver of extensions of $A$ has no loop in $S$ that's to say $Ext^1_A(S,S) = 0$ for every simple $A$-module $S$; while the strong no-loop conjecture is more localized, saying that if the projective dimension of each simple $A$-module $S$ is finite, there is no arrow from $S$ to $S$.

Historically, this conjecture was demonstrated in several cases long before it was formally stated. At the end of 1960 H. Lenzing, proved this conjecture in the case where A is an algebra over an algebraically closed field see [12, Theorem 2.2], it took back up the ideas of Hattori and Stallings on the notion of (the trace of the endomorphism of a projective module). In chronological order, the next result in favor of the conjecture goes back to 1983 when E. L. Green, W. H. Gustafson, and D. Zacharia showed that conjecture is true when the global dimension of algebra is bounded by two see [9]; Their proof consists of a recurrence of the number of isomorphic classes of simple modules. Subsequently, in 1986, K. Fuller and B. Zimmermann-Huisgen demonstrated the conjecture when $(rad(A))^3 = 0$ and when $A$ is a left serial algebra; Their approach uses the matrix Cartan filtered by radical algebra, has demonstrated that the determinant of the matrix thus constructed is one in the case of the finite Global dimension; after K. Igusa, in 1990, proved the conjecture in a case that all algebras of endomorphism of simple modules are separable, the author uses $K$-theory in his proof to point out his result includes that of H. Lenzing since all fields are separable algebras. In 2011 Kiyoshi Igusa, Shiping Liu, and Charles Paquette localized the Lenzing trace function to endomorphisms of modules in $modA$ with the $e$-bounded projective resolution, where $e$ is an idempotent in $A$ and they have shown the strong no-loop conjecture for artinian rings $A$ with $J^2 = 0$ for finite dimensional algebras over an algebraically closed field.

In short, our research work aims to establish the strong no-loop conjecture for artinian rings $A$ with $J^3 = 0$ in the particular case where each simple module having the finite projective dimension whose the projective cover of its first syzygy is canceled by $J^2$, and this is by taking inspiration from the algebra of endomorphisms of a projective $A$-module as well as the Jacobson radical of this one and the characterization of simple and projective $End(P)$-modules with $P$ is a projective $A$- module.

Given a module $M$ in $modA$, we denote by $OmegaM$ the first syzygy, by $pdM$ the projective dimension of $M$ and $\Omega^2(M) = \Omega(\Omega(M))$. We fix a complete set $\{e_1, \ldots, e_n\}$ of orthogonal primitive idempotents in $A$ and let $S_i = Ae_i/Je_i$ the simple $A$-module associated with $e_i$. For convenience, we quote the following well-known result.

**Lemma 1.1.** *Let $A$ be an artinian ring with a short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

*in modA. The following statement holds.*

*(a) $pdN \leq \max\{pdM, pdL+1\}$, and the equality occurs in case $pdM \neq pdL$.*
*(b) $pdL \leq \max\{pdM, pdN-1\}$, and the equality occurs in case $pdM \neq pdN$.*
*(c) $pdM \leq \max\{pdL, pdN\}$, and the equality occurs in case $pdN \neq pdL+1$.*

## 2. Minimal projective dimension

**Lemma 2.1.** *Let $A$ be an artinian ring with a radical cubed zero. Let $M$ be a module in modA of finite projective dimension with,*
*$pd(M) \leq min\{pd(S_1), \ldots, pd(S_n)\}$ and $rad^2(M) = 0$. If $f : P \to M$ is a projective cover of $M$, then :*
*$rad(\Omega M) = rad^2(P)$.*

*Proof.* Let $f : P \to M$ be a projective cover of $M$. Then, $\Omega M \subseteq rad(P)$, and hence:
$rad(\Omega M) \subseteq rad^2(P)$. Since $rad^2(M) = 0$, $rad^2(P) \subseteq \Omega M$.

Suppose that $rad^2(P) \nsubseteq rad(\Omega M)$. Then there exists a maximal submodule $L$ of $\Omega M$ such that $rad^2(P) \nsubseteq L$. Since $rad^3(A) = 0$, $rad^2(P)$ is semi-simple. Thus, $S \nsubseteq L$ where $S$ is some simple submodule of $rad^2(P)$, and consequently, $\Omega M = S \oplus L$. This yields that:

$pd(S) \leq pd(\Omega M) < pdM \leq pd(S)$, a contradiction. The proof of the lemma is completed. $\square$

**Lemma 2.2.** *Let $A$ be an artinian ring with radical cubed zero, and let $S$ be the simple module of minimal projective dimension among the simple modules in mod$A$. If $pdS < \infty$ and $rad^2(P_1) = 0$ with $P_1$ is the projective cover of $\Omega(S)$, then $pdS \leq 1$.*

*Proof.* Suppose that $S$ admits a minimal projective resolution

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0,$$

where $m > 1$ then, $\Omega^2(S) \neq 0$ and, Since $pd(\Omega(S)) < pd(S)$ and, like $A$ has radical cubed zero we have, $rad^2(\Omega(S)) = rad^2(Je) = J^3e = 0$ where $e$ is the idempotent associated with $S$, then by lemma 1.1 and according to the hypothesis of the lemma, $rad(\Omega^2(S)) = rad^2(P_1) = 0$, and therefore $\Omega^2(S)$ is a semisimple module, then $pd(S) \leq pd(\Omega^2(S))$ what is absurd because $pd(\Omega^2(S)) = pd(S) - 2$ . $\square$

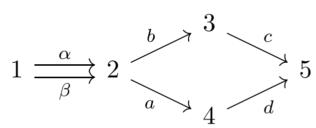
Recall that if $M$ and $N$ are tow modules, by choosen a projective resolution $P_M$ of $M$, then $Ext^n_A(M,N) = H^n(Hom_A(P_*,N))$ is the $n^{th}$ cohomology of the cochain complex of $K$-modules $Hom_A(P_*,N)$ which is:
$... \longrightarrow 0 \longrightarrow 0 \longrightarrow Hom_A(P_0,N) \longrightarrow Hom_A(P_1,N) \longrightarrow Hom_A(P_2,N) \longrightarrow$
...
Where $Hom_A(P_n,N)$ is in degree n and $P_*$ the complex deduced from the resolution $P_M$ by the elimination of $A$-module $M$, see [5, page 427] .

**Theorem 2.3.** *Let $A$ be an artinian ring and $S = Ae/Je$ be a simple $A$-module with e primitive idempotent such that $pd(S) \leq 1$, then:*
$Ext^1_A(S,S) = 0$ .

*Proof.* If S is projective ie $pd(S) = 0$, then the result holds .
If $pd(S) = 1$ and $Ext^1_A(S,S) \neq 0$, then $Hom(Je,S) \neq 0$, therefore $Je$ is projective then $Ae$ is isomorphic to a summand of $Je$ since $Ae$ is a projective cover of $S$, therefore $l(Ae) \leqslant l(Je)$ and since $Je$ is a submodule of $Ae$, then $l(Je) \leqslant l(Ae)$, therefore $l(S) = 0$, a contradiction. $\square$

In what follows we will present an example of an artin algebra with Jacobson cube radical 0 with the condition that any simple module of finite projective dimension has a radical square zero of the cover projective of its first syzygy.

**Example:**
Consider the algebra $A = KQ/I$ given over a field $K$ by a quiver $Q$ and an admissible ideal $I =< bc, ad >$ as following:




Any path in A of length 3 is zero, and therefore $rad^3(KQ/I) = 0$.
In what follows we have a calculation of the projective resolutions of simple modules $(S(i))_{1 \leq i \leq 5}$:

$0 \longrightarrow P(3) \oplus P(4) \longrightarrow P(2)^2 \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$
$0 \longrightarrow P(3) \oplus P(4) \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0$
$0 \longrightarrow P(5) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$
$0 \longrightarrow P(5) \longrightarrow P(4) \longrightarrow S(4) \longrightarrow 0$
$0 \longrightarrow P(5) \longrightarrow P(4) \longrightarrow S(5) \longrightarrow 0$
we have,
$inf(pd(S_i))_{1\leq i\leq 5} = pd(S_2) = pd(S_3) = pd(S_4)$ and $rad^2(P(\Omega(S))) = 0$, where $P(\Omega(S))$ the projectve cover for $\Omega(S)$ for every simple module S, then by lemma1.2 we will have:
$Ext^1_A(S(i), S(i)) = 0$ for $i = 2; i = 3$ and $i = 4$.
This implies that the quiver of extension of $A$ doesn't have a loop at each vertex $S_i$ for $1 \leq i \leq 5$.

## 3. About the Jacobson radical of End(P) with P an A-projective module.

Let P an A-projective module in $modA$, we consider the algebra $E = End_A(P)$ and we denote $J(E)$ the Jacobson radical of the algebra $End_A(P)$ or otherwise $J(E) = rad_{End_A(P)}(End_A(P))$ and according to lemma 1 in [15] , we have:
$J(E) = \{\Phi \in E/\Phi(P) \text{ is a small submodule of } P\}$

**Proposition 3.1.** *For $n \geq 0$ and $P$ an A-projective module in modA we have:*
*(a) $J^n(E) \subseteq Hom(P, rad^n_A(P))$*
*(b) $rad^n_{End(P)}(Hom(P, M)) \subseteq Hom(P, rad^n_A(M))$*

*Proof.* For the first assertion, if $n = 0$ is obvious, we have equality.
And if $\Phi \in J(E)$ we still have by lemma2 in [15], $\Phi(P) \subseteq rad_A(P)$, and consequently:
$\Phi \in Hom(P, rad_A(P))$ and the inclusion is true for $n = 1$, similarly we have:
$J^2(E) = \{\sum g \circ f/f \in J(E) \; et \; g \in J(E)\}$, so:
$f(P) \subseteq rad_A(P)$, and $g(f(P)) \subseteq g(rad_A(P))$, that's to say,
$g \circ f(P) \subseteq g(rad_A(A)P)$ so,
$g \circ f(P) \subseteq rad_A(A)g(P)$ and $g \circ f(P) \subseteq rad^2_A(A)P$ because $g(P) \subseteq rad_A(P)$ from where,
$J^2(E) \subseteq Hom(P, rad^2_A(P))$, so the proposition is true by induction, while the second inclusion comes from the fact that $Hom(P, M)$ is $End(P)$-module on the right and,
$rad_{End(P)}(Hom(P, M)) = Hom(P, M).J(E)$

□

## 4. The no loop conjecture for artinian ring with $J^3 = 0$

Let $A$ be an artinian ring whose $J^3 = 0$ and let $\{P_i = Ae_i\}_{1\leq i\leq n}$ the complete set of non-isomorphic indecomposable projective $A$-modules, for any simple A-module $S$ we assume that $rad^2(P(\Omega(S))) = 0$ with $P(\Omega(S))$ being the projective cover of the first syzygy $\Omega(S)$, let $I$ be the set such that any simple module $S_i$ for $i \in I$ has a finite projective dimension, let $I_1 \subseteq I$ such that $top(P_j) \simeq S_{i_1}$ for $j \in I_1$ where $S_{i_1}$ is the simple $A$-module

of minimal projective dimension among the simple modules in $modA$ for all $i \in I$ .
We consider the algebra $\Gamma_1 = End(\underset{i\in I-I_1}{\oplus} P_i)^{op}$ and we notice $Hom(\underset{i\in I-I_1}{\oplus} P_i, S)$ by $\tilde{S}$ for every simple A-module $S$, then we have the following proposition.

**Proposition 4.1.** *The ring* $\Gamma_1$ *is artinian with:*
*(a)* $rad^3_{\Gamma_1}(\Gamma_1) = 0$.
*(b)* $pd(Hom(\underset{i\in I-I_1}{\oplus} P_i, S)) < \infty$ *for every simple A-module* $S$ *such that,* $pd(S) < \infty$.
*(c)* $rad^2 P(\Omega(\tilde{S})) = 0$ *with* $P(\Omega(\tilde{S}))$ *the projective cover of the first syzygy* $\Omega(\tilde{S})$.

*Proof.* It is easy to verify that $\Gamma_1$ is an artinian ring, we have by Proposition 2.1,
$J^3(\Gamma_1) \subseteq Hom(\Gamma_1, rad^3_A(\underset{i\in I-I_1}{\oplus} P_i))$, as $J^3 = 0$, then $rad^3_{\Gamma_1}(\Gamma_1) = 0$, and since there is an equivalence between the two categories $mod(A)$ and $mod(\Gamma_1)$ see Proposition 2.5 in [5] , then the projective $\Gamma_1$-modules are of the form $Hom(\underset{i\in I-I_1}{\oplus} P_i, Q)$ denoted by $\tilde{Q}$ with $Q$ is a projective A-module and the simple $\Gamma_1$-modules are of the form $Hom(\underset{i\in I-I_1}{\oplus} P_i, S)$ with $S$ a simple A-module, the same if $pd(S) = m$ for a simple A-module S, then by application of the functor $Hom(\underset{i\in I-I_1}{\oplus} P_i, -)$ in the projective resolution

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0$$

we will have,

$$0 \longrightarrow \tilde{P_m} \longrightarrow \cdots \longrightarrow \tilde{P_2} \longrightarrow \tilde{P_1} \longrightarrow \tilde{P_0} \longrightarrow \tilde{S} \longrightarrow 0$$

so $pd_{\Gamma_1}(\tilde{S}) \leq m$.
Because $rad^2_A(P(\Omega(S))) = 0$ and $P(\Omega(\tilde{S})) = P(\tilde{\Omega(S)}) = P(\Omega(S))$ and according to Proposition 2.1(2),
$rad^2_{\Gamma_1}(Hom(\underset{i\in I-I_1}{\oplus} P_i, P(\Omega(S)))) \subseteq Hom(\underset{i\in I-I_1}{\oplus} P_i, rad^2_A(P(\Omega(S))))$ and the third assertion is verified. □

The Simple A-modules $(S_i)_{i\in I-I_1}$ have finite projective dimensions so the $\Gamma_1$-simple modules are also simple, and by the third assertion in proposition 3.1 and by lemma 1.2 the $\Gamma_1$-simple module $\tilde{S_{i_2}}$ of minimal projective dimension among the simple modules in $mod\Gamma_1$ check, $pd_{\Gamma_1}(\tilde{S_{i_2}}) \leq 1$ and $Ext^1_{\Gamma_1}(\tilde{S_{i_2}}, \tilde{S_{i_2}}) = 0$ by theorem 1.3.

**Theorem 4.2.** *If* $\tilde{S_{i_2}}$ *is the* $\Gamma_1$*-simple module of minimal projective dimension among the simple modules of* $mod\Gamma_1$*, then* $Ext^1_A(S_{i_2}, S_{i_2}) = 0$.

*Proof.* $\tilde{S_{i_2}}$ is a simple $\Gamma_1$-module and $\tilde{S_{i_2}} = Hom(\underset{i\in I-I_1}{\oplus} P_i, S_{i_2})$ with,
$S_{i_2} = Ae_{i_2}/Je_{i_2}$ not isomorphic to $S_{i_1}$ with $e_{i_2}$ is the primitive idempotent associated to $S_{i_2}$, then $Ext^1_A(\tilde{S_{i_2}}, \tilde{S_{i_2}}) = 0$ with,
$pd_{\Gamma_1}(\tilde{S_{i_2}}) = 0$ *or* 1 so:
- If $\tilde{S_{i_2}}$ is projective, then $S_{i_2}$ is projective and $Ext^1_A(S_{i_2}, S_{i_2}) = 0$ .

• If $Ext^1_A(S_{i_2}, S_{i_2}) \neq 0$, then $Hom(Je_{i_2}, S_{i_2}) \neq 0$ and the exact sequence: $0 \longrightarrow Je_{i_2} \longrightarrow P_{i_2} \longrightarrow S_{i_2} \longrightarrow 0$ will not be split, by the same $pd_A(S_{i_2}) \geq 2$ because $i_2 \notin I_1$ and by application of the functor $Hom(\underset{i\in I-I_1}{\oplus} P_i, -)$, we will have the exact sequence:
$0 \longrightarrow Hom(\underset{i\in I-I_1}{\oplus} P_i, Je_{i_2}) \longrightarrow Hom(\underset{i\in I-I_1}{\oplus} P_i, P_{i_2}) \longrightarrow Hom(\underset{i\in I-I_1}{\oplus} P_i, S_{i_2}) \longrightarrow 0$, and like,
$Hom(\underset{i\in I-I_1}{\oplus} P_i, Je_{i_2})$ is not projective, so : $pd_{\Gamma_1}(Hom(\underset{i\in I-I_1}{\oplus} P_i, S_{i_2}) > 1$ i.e. $pd_{\Gamma_1}(\tilde{S_{i_2}}) > 1$ what is absurd so: $Ext^1_A(S_{i_2}, S_{i_2}) = 0$. □

Thus by induction and in the same way we consider the artin algebras: $\Gamma_k = End(\underset{i\in I-I_1\cup I_2\cup I_3...\cup I_k}{\oplus} P_i)^{op}$ where $I_k \subseteq I$ such as:
$top(P_i) \simeq S_{i_{k+1}}$ for all index $i \in I - I_1 \cup I_2 \cup I_3 ... \cup I_k$ and $S_{i_{k+1}}$ be a simple module of minimal projective dimension among the simple modules in $mod\Gamma_k$ for all $i \in I - I_1 \cup I_2 \cup I_3 ... \cup I_k$ , and $Ext^1_A(S_{i_k}, S_{i_k}) = 0$, then we have the following theorem.

**Theorem 4.3.** *Let A be an artinian ring with $J^3 = 0$, if $rad^2(P(\Omega(S))) = 0$ for every simple module S, then $Ext^1_A(S, S) = 0$.*

It follows from this theorem the new proof of the strong no-loop conjecture for Artinian rings with $J^2$=0 this subject was treated by [12] in finite dimension drawing inspiration from the results of cyclic homology and Hattori and Stalling and the Hochschild homological group since the conditions of the theorem are verified.

**Corollary 4.4.** *Let A be an artinian ring with radical cubed zero such that the projective cover of $rad(A)$ is of Loewy length two.*
*If $gdim(A)$ is finite, then $Ext^1_A(S, S) = 0$ for every simple module S.*

*Proof.* If $rad^2(P(rad(A))) = 0$, then $rad^2(P(\Omega(S))) = 0$ for all simple A-modules indeed:
$A/rad(A) = \underset{i\in J}{\oplus} S_i^{a_i}$ where the $(S_i)_{i\in J}$ are all the A non-isomorphic simple modules and as we have the exact sequence:

$$0 \longrightarrow rad(A) \longrightarrow A \longrightarrow A/rad(A) \longrightarrow 0$$

so,
$\Omega(A/rad(A)) = rad(A)$ and if $rad^2(P(rad(A))) = 0$, then,
$rad^2(P(\Omega(A/rad(A)))) = 0$ where from $rad^2(P(\Omega(S_i))) = 0$ and according to the theorem 3.4, we have $Ext^1_A(S, S) = 0$. □

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